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The Taylor column problem

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We consider here flow past an obstacle on the lower of two rotating horizontal planes which bound a viscous fluid. It is found that when the Taylor number is large viscous effects are confined to Ekman boundary layers on the solid surfaces and to a free shear layer coincident with the vertical cylinder circumscribing the bottom obstacle. The flow in the main body of the fluid outside the cylinder proves to be two-dimensional with zero relative vorticity, while inside the cylinder the fluid is stagnant in the rotating frame. The shear layer provides a continuous transition between the exterior and interior of the cylinder, and in addition provides a means by which fluid from the Ekman layers on the horizontal planes is exchanged with fluid outside the Ekman layers and exterior to the circumscribing cylinder. The predicted flow proves to be in agreement with many of the experimental results.

1. Introduction

When a solid body moves horizontally with low Rossby number in a fluid rotating about a vertical axis, the vertical cylinder circumscribing the body separates regions of dissimilar velocity distribution. The fluid inside the circumscribing cylinder moves with the body while the fluid outside the cylinder flows around it in a two-dimensional pattern. This result was discovered experimentally by Taylor (1923) and the phenomenon is often called a Taylor column.

Certain features of the experiment can be explained on the basis of the Taylor–Proudman theorem, which states that steady geostrophic motions are independent of distance along the rotation vector. For a vertical rotation vector the presence of bounding horizontal planes suppresses vertical motions. Therefore the flow inside a vertical cylinder circumscribing a three-dimensional body must be directed along the contours of the body. The absence of flow through the circumscribing cylinder is thereby explained, but an argument based on the Taylor–Proudman theorem alone cannot account for the stagnation relative to the body inside the circumscribing cylinder observed by Taylor.

A number of authors (e.g. Stewartson 1953) have studied the motion of bodies in a rotating fluid using an inviscid time-dependent model. The ultimate steady-state solution for this problem when the motion is at right angles to the axis of rotation is in marked disagreement with experiment. It appears that the limit $\nu \to 0$, time $\to \infty$, is non-uniform, and that an inviscid theory is valid for small times only.

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We consider here steady viscous flow past a bottom obstacle in a rotating fluid of finite depth. The model is simple but is believed to contain the features necessary for the phenomenon. This is confirmed by the solution, which is in satisfactory agreement with Taylor's observations.

2. Formulation

Let $\mathbf{x}_* = (x_*, y_*, z_*)$ be the position vector in a co-ordinate system rotating with angular velocity $\Omega\hat{\mathbf{z}}_*$, $\mathbf{q}^*$ the velocity vector, p_* the pressure, ρ the density, and ν the kinematic viscosity. The fluid is bounded by rigid horizontal planes a distance D apart at rest in the rotating frame, with an obstacle on the lower plane with circular contours and finite horizontal extent L. At $x_*^2 + y_*^2 = \infty$ the flow in the main body of the fluid is uniform with magnitude c in the positive x_* direction.

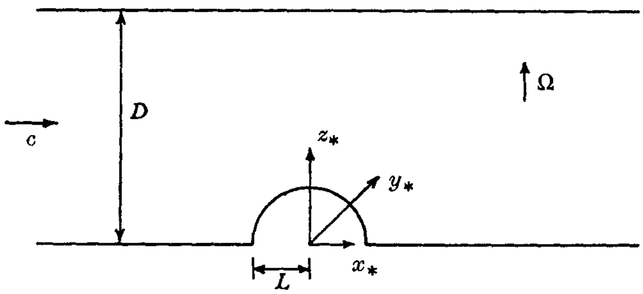

FIGURE 1. Configuration.

If we introduce dimensionless unstarred variables through

$$\mathbf{x}_* = L\mathbf{x}, \quad \mathbf{q}_* = c\mathbf{q}, \quad p_*/\rho = \tfrac{1}{2}\Omega^2(x_*^2 + y_*^2) + 2\Omega c L p,$$

the steady-state dimensionless equations of motion become

$$\epsilon(\mathbf{q}\,.\,\nabla)\,\mathbf{q} + \hat{\mathbf{z}} \times \mathbf{q} + \nabla p = (1/2R)\,\nabla^2\mathbf{q}, \tag{2.1}$$

$$\nabla\,.\,\mathbf{q} = 0, \tag{2.2}$$

where $\epsilon = c/(2\Omega L)$ is the Rossby number and $R = \Omega L^2/\nu$ a Taylor number. Letting $d = D/L$ be the dimensionless separation distance and $z = b(r)$ the equation of the bottom obstacle, where $r = (x^2+y^2)^{\frac{1}{2}}$, we have as boundary conditions $\mathbf{q} = 0$ at $z = d$, at $z = 0$ for $r \geqslant 1$, and at $z = b(r)$ for $r \leqslant 1$. We consider only the case of small ϵ and large R, and assume that the non-linear terms can be neglected. The problem then reduces to solving the linear equations

$$\hat{\mathbf{z}} \times \mathbf{q} + \nabla p = (1/2R)\,\nabla^2\mathbf{q}, \tag{2.3}$$

and (2.2) subject to the condition $\mathbf{q} = 0$ at solid surfaces.

It proves convenient to use cylindrical co-ordinates (r, θ, z) with velocity components (u, v, w). Equations (2.3) and (2.2) in component form are given by

$$-v + \frac{\partial p}{\partial r} = \frac{1}{2R}\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right), \tag{2.4}$$

$$u + \frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{1}{2R}\left(\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right), \tag{2.5}$$

$$\frac{\partial p}{\partial z} = \frac{1}{2R}\nabla^2 w, \tag{2.6}$$

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0. \tag{2.7}$$

3. The Ekman layers

The awkward geometry precludes an exact treatment of the linear problem, but an approximate solution can be obtained through the use of singular perturbation theory. A necessary first step is the calculation of the normal velocity at the edges of the Ekman layers on the solid boundaries.

For the boundary $z = b(r)$, set $\mathbf{q} = \mathbf{q}_I + \mathbf{q}_B$, $p = p_I + p_B$, where the boundary-layer parts $\mathbf{q}_B$ and p_B are to be exponentially small away from the boundary. Let $\zeta = z - b(r)$ so that (r, θ, ζ) is an oblique co-ordinate system with $\zeta = 0$ at the boundary, and let $t = R^{\frac{1}{2}}\zeta$. The Laplacian is $R(1+b'^2)(\partial^2/\partial t^2) + O(R^{\frac{1}{2}})$, and the boundary-layer equations become

$$-v_B + \frac{\partial p_B}{\partial r} - R^{\frac{1}{2}} b' \frac{\partial p_B}{\partial t} = \tfrac{1}{2}(1+b'^2)\frac{\partial^2 u_B}{\partial t^2}, \tag{3.1}$$

$$u_B + \frac{1}{r}\frac{\partial p_B}{\partial \theta} = \tfrac{1}{2}(1+b'^2)\frac{\partial^2 v_B}{\partial t^2}, \tag{3.2}$$

$$R^{\frac{1}{2}}\frac{\partial p_B}{\partial t} = \tfrac{1}{2}(1+b'^2)\frac{\partial^2 w_B}{\partial t^2}, \tag{3.3}$$

$$\frac{1}{r}\frac{\partial (r u_B)}{\partial r} + \frac{1}{r}\frac{\partial v_B}{\partial \theta} + R^{\frac{1}{2}}\left(\frac{\partial w_B}{\partial t} - b'\frac{\partial u_B}{\partial t}\right) = 0. \tag{3.4}$$

Now let $$F = \frac{1}{\sqrt{(1+b'^2)}}(w - b'u), \quad G = \frac{1}{\sqrt{(1+b'^2)}}(u + b'w),$$

be the velocity components normal to, and along a generator of, $\zeta = 0$. From (3.3) and (3.4), p_B and F_B are $O(R^{-\frac{1}{2}})$ and the boundary-layer equations to lowest order are

$$-v_B = \tfrac{1}{2}(1+b^{\frac{1}{2}})^{\frac{3}{2}}(\partial^2 G_B/\partial t^2), \tag{3.5}$$

$$G_B = \tfrac{1}{2}(1+b'^2)^{\frac{3}{2}}(\partial^2 v_B/\partial t^2), \tag{3.6}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r}{\sqrt{(1+b'^2)}}G_B\right] + \frac{1}{r}\frac{\partial v_B}{\partial \theta} + R^{\frac{1}{2}}\sqrt{(1+b'^2)}\frac{\partial F_B}{\partial t} = 0. \tag{3.7}$$

The solution of (3.5) and (3.6) satisfying

$$\mathbf{q}_I(r,\theta,0) + \mathbf{q}_B(r,\theta,0) = 0$$

is $$v_B = -e^{-\tau}[v_I(r,\theta,0)\cos\tau - G_I(r,\theta,0)\sin\tau], \tag{3.8}$$

$$G_B = -e^{-\tau}[G_I(r,\theta,0)\cos\tau + v_I(r,\theta,0)\sin\tau], \tag{3.9}$$

where $$\tau = (1+b'^2)^{-\frac{3}{4}}t.$$

Substitution into the continuity equation, integration to get F_B, and application of the boundary condition yields

$$F_I = \frac{1}{2R^{\frac{1}{2}}r\sqrt{(1+b'^2)}}\left\{(1+b'^2)^{-\frac{3}{4}}\frac{\partial}{\partial \theta}(v_I - G_I) + \frac{\partial}{\partial r}[r(1+b'^2)^{\frac{1}{4}}(v_I + G_I)]\right\} \tag{3.10}$$

at $\zeta = 0$.

Changing back to cylindrical polar co-ordinates, we obtain

$$w_I - b'u_I = \frac{1}{2R^{\frac{1}{2}}r}\left\{(1+b'^2)^{\frac{3}{4}}\frac{\partial}{\partial\theta}\left(v_I - \frac{u_I + b'w_I}{\surd(1+b'^2)}\right) + \left(\frac{\partial}{\partial r} + b'\frac{\partial}{\partial z}\right)\left[r(1+b'^2)^{\frac{1}{4}}\left(v_I + \frac{u_I + b'w}{\surd(1+b'^2)}\right)\right]\right\} \quad (3.11)$$

at $z = b(r)$. Setting $b = 0$ yields the boundary condition at $z = 0$, $r > 1$,

$$w_I = \frac{1}{2R^{\frac{1}{2}}}\left\{\hat{\mathbf{z}}\,.\,\nabla\times\mathbf{q}_I - \frac{\partial w_I}{\partial z}\right\}. \quad (3.12)$$

Equations (3.5), (3.6), and (3.7) are symmetrical under the transformation $t \to -t$, $F_B \to -F_B$, so at $z = d$

$$w_I = -\frac{1}{2R^{\frac{1}{2}}}\left\{\hat{\mathbf{z}}\,.\,\nabla\times\mathbf{q}_I - \frac{\partial w_I}{\partial z}\right\}. \quad (3.13)$$

These three equations provide the boundary conditions for the interior problem. It may be noted that when b' is $O(R^{\frac{1}{2}})$ the boundary-layer thickness is no longer small and the boundary-layer calculation invalid. With little loss of generality, we consider here only obstacles with slope of order unity or smaller.

4. The interior problem

We consider below the flow outside of the Ekman layers, and accordingly drop the subscript on $\mathbf{q}_I$. We assume initially that in the main body of the fluid $\mathbf{q}$ is a smooth function of $\mathbf{x}$. Then to lowest order the momentum equation becomes

$$\hat{\mathbf{z}}\times\mathbf{q} + \nabla p = 0, \quad (4.1)$$

the curl of which is the Taylor–Proudman theorem,

$$\partial\mathbf{q}/\partial z = 0. \quad (4.2)$$

In the region $0 \leqslant r < 1$, $\mathbf{q}$ is found by solving

$$\frac{\partial(ru)}{\partial r} + \frac{\partial v}{\partial\theta} = 0, \quad (4.3)$$

$$w = -\frac{1}{2R^{\frac{1}{2}}r}\left[\frac{\partial(rv)}{\partial r} - \frac{\partial u}{\partial\theta}\right], \quad (4.4)$$

$$w - b'u = \frac{1}{2R^{\frac{1}{2}}r}\left\{(1+b'^2)^{\frac{3}{4}}\frac{\partial}{\partial\theta}\left(v - \frac{u+b'w}{\surd(1+b'^2)}\right) + \frac{\partial}{\partial r}\left[r(1+b'^2)^{\frac{1}{4}}\left(v + \frac{u+b'w}{\surd(1+b'^2)}\right)\right]\right\}, \quad (4.5)$$

and in the region $r > 1$ by solving (4.3), (4.4), and

$$w = \frac{1}{2R^{\frac{1}{2}}r}\left[\frac{\partial(rv)}{\partial r} - \frac{\partial u}{\partial\theta}\right]. \quad (4.6)$$

We wish first to show that any $\mathbf{q}$ which is continuous and axially symmetrical over the bottom obstacle must vanish. In the case of axial symmetry, $u = 0$ for boundedness at $r = 0$. As w is $O(R^{-\frac{1}{2}})$, (4.5) becomes

$$w = \frac{1}{2R^{\frac{1}{2}}r}\frac{d}{dr}[r(1+b'^2)^{\frac{1}{2}}v] \tag{4.7}$$

to lowest order, and this combines with (4.4) to yield

$$d\{r[1+(1+b'^2)^{\frac{1}{2}}]v\}/dr = 0. \tag{4.8}$$

Therefore $v = w = 0$ by the boundedness condition. If an ordinary perturbation expansion

$$\mathbf{q} = \mathbf{q}(0) + R^{-\frac{1}{2}}\mathbf{q}^{(-\frac{1}{2})} + R^{-1}\mathbf{q}^{(-1)} + \ldots \tag{4.9}$$

is introduced, it is easily verified that if $\mathbf{q}^{(0)} = 0$ the rest of the terms in the series also vanish.

Now if for some $r_0 < 1$, b' is $O(R^0)$ on $r_0 < r < 1$, then

$$u^{(0)} = w^{(0)} = \partial v^{(0)}/\partial\theta = 0$$

in this region. $\mathbf{q}^{(0)}$ is axially symmetric, must therefore be axially symmetric for $r < r_0$, and hence vanishes on $0 \leqslant r < 1$. The rest of the terms in the series also vanish, and $\mathbf{q} = 0$ over the bottom obstacle.

This result is not obtained if b' is small everywhere, for then $\mathbf{q}$ is not directed along the depth contours and need not be axially symmetric. For dealing with this case, let $b' = -R^{-\frac{1}{2}}B(r)$ and regard B as being of order unity. Introducing (4.9), we obtain for the lowest order equations $w^{(0)} = 0$ and

$$\frac{\partial(ru^{(0)})}{\partial r} + \frac{\partial v^{(0)}}{\partial\theta} = 0, \tag{4.10}$$

$$\frac{1}{r}\left[\frac{\partial(rv^{(0)})}{\partial r} - \frac{\partial u^{(0)}}{\partial\theta}\right] - B(r)\,u^{(0)} = 0. \tag{4.11}$$

Eliminating $v^{(0)}$ and setting

$$ru^{(0)} = \mathrm{Re}\,[f_n(r)\,e^{in\theta}],$$

yields

$$r^2f_n'' + rf_n' - (n^2 - inB)f_n = 0, \tag{4.12}$$

which for $n \neq 0$ has a solution behaving like $r^{1+\delta}$ at the origin, δ being positive. If b' is $O(R^{-\frac{1}{2}})$ there is no Taylor column phenomenon.

For $r > 1$, we have

$$w = \tfrac{1}{2}R^{-\frac{1}{2}}\hat{\mathbf{z}}\,.\,\nabla\times\mathbf{q} = -\tfrac{1}{2}R^{-\frac{1}{2}}\hat{\mathbf{z}}\,.\,\nabla\times\mathbf{q} = 0 \tag{4.13}$$

and the flow in this region is irrotational and two-dimensional. At $r = \infty$

$$u = \cos\theta, \quad v = -\sin\theta, \quad w = 0,$$

in the main body of the fluid. Therefore the horizontal velocity is given by the irrotational two-dimensional solution

$$u = \mathrm{Re}\,[(1 - A_1/r^2)\,e^{i\theta}], \tag{4.14}$$

$$v = \mathrm{Re}\,[A_0/r + i(1 + A_1/r^2)\,e^{i\theta}], \tag{4.15}$$

where A_0 and A_1 are constants, and the axially symmetric mode represents a circulation. If $\mathbf{q} = 0$ for $r < 1$, there must be a vertical shear layer at $r = 1$ to provide a smooth transition between the two flow régimes. The shear layer solution determines A_0 and A_1. If b' is small, $\mathbf{q}$ does not vanish over the bottom obstacle, and the constants are determined by matching.

It should be noted that the details of the obstacle shape have no bearing on whether or not a Taylor column is formed, the only restriction being that the slope of the obstacle is of order unity somewhere within the circumscribing cylinder. The influence of bottom slope on the phenomenon can be deduced from a physical argument based on the result for the normal velocity component at the edges of the Ekman layers. For constant slope

$$F = \frac{(1+b'^2)^{\frac{1}{4}}}{2R^{\frac{1}{2}}}\left\{\hat{\mathbf{n}}\,.\,\nabla\times\mathbf{q} - \frac{1}{r}\,\hat{\mathbf{n}}\,.\,\nabla F\right\}$$

at $z = b$, where $\hat{\mathbf{n}}$ is the outward normal to the surface. If a horizontal stream is forced over the bottom obstacle, there is a flux into the Ekman layer of magnitude b' which induces a relative vorticity of magnitude $b'R^{\frac{1}{2}}$. For b' of order unity, the curvature of the streamlines is so large that the fluid flows around rather than over the obstacle. The time development of a Taylor column would probably consist of the formation of a closed streamline pattern over the obstacle followed by a decay of this motion through the spin-down mechanism described by Greenspan & Howard (1963). If the bottom slope is small, the induced vorticity is not large enough to cause a closed streamline pattern over the obstacle.

A different vorticity argument, based on an inviscid inertial theory, has been developed by Hide (1961). Hide uses the potential vorticity theorem, which states that

$$\frac{1+\epsilon\hat{\mathbf{z}}\,.\,\nabla\times\mathbf{q}}{d-b}$$

is constant on streamlines. A streamline originating at $r = \infty$ acquires a relative vorticity equal to $-b/(\epsilon d)$ as it passes over a bottom obstacle. For small Rossby number the curvature of the streamlines is large and a closed streamline pattern results. An inviscid theory can be useful in describing the initial states of the formation of a Taylor column even if the theory is invalid for large times, and Hide's mechanism would seem to be correct when the Rossby number is sufficiently large that the non-linear accelerations cannot be neglected.

As a final remark on the results obtained thus far, it is easily verified that the boundary conditions for w at the plane boundaries are unchanged if the planes move parallel to themselves with constant velocity. Therefore the results are equally valid for an obstacle moving in a fluid at rest at infinity, the only difference being that the Ekman layers on the plane boundaries vanish at $r = \infty$.

5. The shear layer

The shear layer which provides the transition for removing discontinuities in $\mathbf{q}$ is similar to the vertical boundary layers in axially symmetric flow discussed in the literature, most recently by Greenspan & Howard (1963). In treating the flow for z and $(d-z) > R^{-\frac{1}{2}}$ the boundary conditions derived previously for the normal velocity component at the edges of the Ekman layers may be used. The approximate form of these boundary conditions appropriate for a shear layer at $r = 1$ is

$$w = -\frac{1}{2R^{\frac{1}{2}}}\frac{\partial v}{\partial r} \quad \text{at} \quad z = d, \tag{5.1}$$

and

$$w - b'u = \frac{1}{2R^{\frac{1}{2}}}\frac{\partial}{\partial r}[(1+b'^2)^{\frac{1}{4}} v] \quad \text{at} \quad z = b(r), \tag{5.2}$$

where $b = 0$ for $r \geqslant 1$. To lowest order the lower boundary condition can be applied at $z = 0$ since $b(1) = 0$. The equations of the shear layer are obtained by retaining only the most highly differentiated terms in r in the equations of motion written in polar co-ordinates and setting $r = 1$ wherever it occurs, and are given by

$$\frac{\partial w}{\partial z} = -\frac{1}{2R}\frac{\partial^3 v}{\partial r^3}, \tag{5.3}$$

$$\frac{\partial v}{\partial z} = \frac{1}{2R}\frac{\partial^3 w}{\partial r^3}, \tag{5.4}$$

$$\frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0. \tag{5.5}$$

We will use a boundary-layer theory of matching type and require that $\mathbf{q} = 0$ for $(r-1)$ negative and order unity, and $w = 0$ and

$$u = \mathrm{Re}\,[(1 - A_1/r^2)\, e^{i\theta}], \tag{4.14 bis}$$

$$v = \mathrm{Re}\,[A_0/r + i(1 + A_1/r^2)\, e^{i\theta}], \tag{4.15 bis}$$

for $(r-1)$ positive and order unity.

Eliminating v between (5.3) and (5.4) yields

$$\frac{\partial^6 w}{\partial r^6} + 4R^2 \frac{\partial^2 w}{\partial z^2} = 0 \tag{5.6}$$

which implies that

$$\zeta = R^{\frac{1}{3}}(r-1)$$

is the scaled boundary-layer co-ordinate if $(\partial^2 w/\partial z^2)$ is of the same order in R as w. However, the shear layer cannot exist if $R^{-\frac{1}{3}}$ is the only scale of motion. This is because v and w are of the same order of magnitude in a shear layer of thickness $R^{-\frac{1}{3}}$ and w vanishes at $z = d$ and $z = 0$ with an error of order $R^{-\frac{1}{6}}$. The equation for w is then homogeneous with homogeneous boundary conditions.

Another scale is obtained by requiring the right and left sides of (5.1) and (5.2) to have the same order of magnitude. If the right and left sides of (5.4) are also of the same order in R, the scale of motion is $R^{-\frac{3}{8}}$, the ratio of v to w is of order $R^{\frac{1}{8}}$, and (5.3) is not of boundary-layer character. The remaining possibility is for the two sides of (5.3) to be in balance. The resulting length scale is $R^{-\frac{1}{4}}$, u and w are smaller than v by a factor of $R^{-\frac{1}{4}}$, and the equations of the $R^{-\frac{1}{4}}$ layer are

$$\frac{\partial w}{\partial z} = -\frac{1}{2R}\frac{\partial^3 v}{\partial r^3}, \tag{5.7}$$

$$\partial v/\partial z = 0, \tag{5.8}$$

$$\frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} = 0. \tag{5.9}$$

Integrating (5.7) over z and applying the upper boundary condition, we obtain

$$w = \frac{(d-z)}{2R}\frac{\partial^3 v}{\partial r^3} - \frac{1}{2R^{\frac{1}{2}}}\frac{\partial v}{\partial r}, \tag{5.10}$$

and application of the lower boundary condition yields

$$\frac{d}{2R}\frac{\partial^3 v}{\partial r^3} - \frac{1}{2R^{\frac{1}{2}}}\frac{\partial}{\partial r}\{[1+(1+b'^2)^{\frac{1}{2}}]\,v\} - b'u = 0. \tag{5.11}$$

If the motion is axially symmetric, $u = 0$ to lowest order from (5.9) and the boundary condition at the inner edge of the shear layer. Integrating (5.11) and applying the condition at the inner edge of the shear layer yields

$$\frac{d^2 v}{dr^2} - \frac{1+(1+b'^2)^{\frac{1}{2}}}{2R^{\frac{1}{2}}d}\,v = 0, \tag{5.12}$$

which has $v = 0$ as the only solution which does not grow exponentially as $|r-1|$ becomes of order unity. Therefore there is no axially symmetric mode. Now let

$$u = \mathrm{Re}\,[f(r)\,e^{i\theta}] \tag{5.13}$$

so that

$$v = \mathrm{Re}\,[f' i e^{i\theta}], \tag{5.14}$$

$$w = \mathrm{Re}\left\{\left[\frac{(d-z)}{2R}f^{\mathrm{iv}} - \frac{1}{2R^{\frac{1}{2}}}f''\right] i\,e^{i\theta}\right\}, \tag{5.15}$$

and (5.11) becomes

$$(d/2R)f^{\mathrm{iv}} - \tfrac{1}{2}R^{-\frac{1}{2}}\{[1+(1+b'^2)^{\frac{1}{2}}]f'\}' + ib'f = 0. \tag{5.16}$$

We consider first the case $b'(1-) \neq 0$, corresponding to a discontinuity in bottom slope at $r = 1$, and let $b'(1-) = -\tan\beta$. Then to lowest order

$$(1+b'^2)^{\frac{1}{2}} = H(r-1) + \frac{1}{\sqrt{(\cos\beta)}}H(1-r) \tag{5.17}$$

in the shear layer, where H is equal to 1 for positive argument and 0 for negative argument. Equation (5.16) has constant coefficients for $|r-1| > 0$ and

$$f'''(1+) - f'''(1-) = \frac{\sqrt{(\cos\beta)}-1}{d\sqrt{(\cos\beta)}}R^{\frac{1}{2}}f'(1). \tag{5.18}$$

Letting $\eta = R^{\frac{1}{4}}(r-1)$ be the scaled shear-layer co-ordinate, we obtain

$$\frac{d^4 f}{d\eta^4} - \frac{2}{d}\frac{d^2 f}{d\eta^2} = 0 \quad \text{for} \quad \eta > 0 \tag{5.19}$$

and

$$\frac{d^4 f}{d\eta^4} - \frac{\sqrt{(\cos\beta)}+1}{d\sqrt{(\cos\beta)}}\frac{d^2 f}{d\eta^2} - \frac{2i}{d}\tan\beta f = 0 \quad \text{for} \quad \eta < 0, \tag{5.20}$$

with f and its first two derivatives continuous at $\eta = 0$ and with

$$\frac{d^3 f(0+)}{d\eta^3} - \frac{d^3 f(0-)}{d\eta^3} = \frac{\sqrt{(\cos\beta)}-1}{d\sqrt{(\cos\beta)}}\frac{df(0)}{d\eta}. \tag{5.21}$$

The solution for f is

$$f = \gamma\{H(-\eta)[B_1 e^{s_1\eta} + B_2 e^{s_2\eta}] + H(\eta)[B_3 e^{-s_3\eta} + B_4 + \eta]\}, \tag{5.22}$$

where $s_1, s_2 = (2d)^{-\frac{1}{2}}(\cos\beta)^{-\frac{1}{4}}\{1+(\cos\beta)^{\frac{1}{2}} \pm [(1+(\cos\beta)^{\frac{1}{2}})^2 + 8id\sin\beta]^{\frac{1}{2}}\}^{\frac{1}{2}}$ are the solutions with positive real parts of

$$s^4 - \frac{\sqrt{(\cos\beta)}+1}{d\sqrt{(\cos\beta)}} s^2 - \frac{2i}{d}\tan\beta = 0,$$

$s_3 = (2/d)^{\frac{1}{2}}$, γ is a constant to be determined from the conditions at the outer edge of the shear layer, and

$$B_1 = \frac{1}{s_1 Q}\{s_3[s_1^2 - s_2 s_3 - s_3^2]\},$$

$$B_2 = \frac{1}{s_2 Q}\{s_3[s_3^2 + s_1 s_3 - s_2^2]\},$$

$$B_3 = \frac{(s_1 - s_2)}{s_3 Q}\{s_1^2 + s_1 s_2 + s_2^2 - s_3^2\},$$

$$B_4 = \frac{(s_1 - s_2)}{s_1 s_2 s_3 Q}\{s_3^4 + s_3^3(s_1+s_2) + s_1 s_2[2s_3^2 - (s_1^2 + s_1 s_2 + s_2^2)]\},$$

where $$Q = s_3(s_1^2 - s_2^2) + (s_1^3 - s_2^3).$$

For large positive η,

$$u = \operatorname{Re}\{[\gamma(B_4 + R^{\frac{1}{4}}(r-1))]e^{i\theta}\}, \quad v = \operatorname{Re}[\gamma R^{\frac{1}{4}} i e^{i\theta}],$$

while for small positive $(r-1)$ (4.14) and (4.15) become

$$u = \operatorname{Re}\{[1 - A_1 + 2A_1(r-1)]e^{i\theta}\}, \quad v = \operatorname{Re}\{[(1 + A_1 - 2A_1(r-1)]ie^{i\theta}\},$$

whence $$\gamma = 2R^{-\frac{1}{4}} + O(R^{-\frac{1}{2}}), \quad A_1 = 1 - 2R^{-\frac{1}{4}}B_4 + O(R^{-\frac{1}{2}}).$$

Therefore, with terms of order $R^{-\frac{1}{2}}$ neglected, $w = 0$ and

$$u = \operatorname{Re}\{[(1 - 1/r^2) + 2R^{-\frac{1}{4}}B_4/r^2]e^{i\theta}\}, \tag{5.23}$$

$$v = \operatorname{Re}\{[(1 + 1/r^2) - 2R^{-\frac{1}{4}}B_4/r^2]ie^{i\theta}\}, \tag{5.24}$$

in the main body of the fluid for $r > 1$. To lowest order the flow pattern is the same as for potential flow around a cylinder, but a small radial velocity at $r = 1$ is caused by a convergence or divergence in the shear layer.

The solution for w as calculated from (5.15) and (5.22) has a delta function singularity at $r = 1$ which is removed in a shear layer of thickness $R^{-\frac{1}{3}}$. The v and w components induced in this shear layer are of order $R^{-\frac{1}{4}}$ and the u component of order $R^{-\frac{1}{2}}$. The form of these solutions is similar but not identical to that of the solutions given by Greenspan & Howard (1963).

The only other case we will consider is that of $b'(1-) = 0$ and $b''(1-) = \lambda > 0$, corresponding to a bottom with continuous slope and discontinuous curvature at $r = 1$. In the shear layer

$$b' = \lambda(r-1)H(1-r) = R^{-\frac{1}{4}}\lambda\eta H(-\eta), \tag{5.25}$$

and the equation for f for $\eta < 0$ is

$$\frac{d^4 f}{d\eta^4} - \frac{2}{d}\frac{d^2 f}{d\eta^2} - \tfrac{1}{2}\lambda^2 R^{-\frac{1}{2}}\eta\frac{df}{d\eta} + \frac{2i\lambda R^{-\frac{1}{4}}}{d}\eta f = 0, \tag{5.26}$$

instead of (5.20). A solution can be obtained in the form of contour integrals which can be evaluated for large R by means of the method of stationary phase. The result is that (5.26) has a rapidly varying solution of the form $\exp(s_3\eta)$ and a slowly varying solution in the form of an Airy function,

$$(-\eta)^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}[\tfrac{2}{3}\lambda^{\frac{1}{2}} R^{-\frac{1}{8}} e^{\frac{1}{4}i\pi}(-\eta)^{\frac{3}{2}}]. \tag{5.27}$$

The slowly varying solution has a length scale $R^{-\frac{1}{8}}$ and satisfies

$$\frac{d^2 f}{d\eta^2} - i\lambda R^{-\frac{1}{4}}\eta f = 0, \tag{5.28}$$

which can be obtained by neglecting viscosity entirely in the main body of the fluid and retaining it only in the boundary conditions (5.1) and (5.2). The shear layer is thicker in this case than for a bottom with discontinuous slope at $r = 1$.

There are a number of points of interest connected with the shear layer. In addition to providing a transition between the potential flow region and the Taylor column, the shear layer serves as a means by which fluid is transferred from the Ekman layers at $z = 0$ and $z = d$ to the potential flow region. From (5.1) and (5.2), the transport from the Ekman layer parts of the shear layer into the interior part of the shear layer is given by

$$\tfrac{1}{2}R^{-\frac{1}{2}}\,\mathrm{Re}\,[2i\, e^{i\theta}] = -R^{-\frac{1}{2}}\sin\theta,$$

which is equal to the radial transport into the shear layer in each of the Ekman layers on the plane boundaries. This fluid is no doubt returned to the main body of the fluid by means of an order $R^{-\frac{1}{2}}$ radial velocity at the outer edge of the shear layer.

Equation (5.22) and the formulae following it show that the effect of taking d to be very large is to thicken the shear layer, and in the limit $d \to \infty$ the shear layer approach is invalid. A likely explanation is that the limit $d \to \infty$, time $\to \infty$ is non-uniform and that the character of the flow is different for large d than that given above.

If b' is small in the neighbourhood of $r = 1$, the shear layer is thicker than if $b'(1-)$ is of order unity, and motions extend further in over the bottom obstacle. If b' is small everywhere, there is no Taylor column and (5.22) represents the form of the solution for small $(r-1)$ obtained by matching the flow over the obstacle to the potential flow.

6. Concluding remarks

An estimate of how small the Rossby number must be for the non-linear accelerations to be neglected can be made by calculating the magnitude of the neglected terms. The most stringent restriction on the magnitude of ϵ comes from a comparison of the θ components of the non-linear and Coriolis accelerations in the Ekman layer on the bottom obstacle. The former of these is of order $\epsilon R^{\frac{1}{4}}$ and the latter of order $R^{-\frac{1}{4}}$, so the linearization is internally consistent only if $\epsilon R^{\frac{1}{2}} < 1$, a severe restriction.

The agreement with Taylor's experimental results is fairly good. In the large R limit the predicted flow in the main body of the fluid consists of stagnation for

$r < 1$ and two-dimensional irrotational flow without circulation for $r > 1$, with no radial flow through $r = 1$. These features of the flow agree with Taylor's observations. However, Taylor has observed that one branch of the dividing streamline leaves the cylinder $r = 1$ and breaks up into eddies. There is no indication of this in the present theory.

The agreement with recent experiments by Ibbetson (1963) is less satisfactory. Ibbetson finds that the flow is highly unstable when the bottom obstacle is a cone or hemisphere, and he has been unable to find a Taylor column for any bottom obstacle but a right circular cylinder. His results contradict Taylor's and limit the usefulness of the present theory.

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